STRUCTURE IN THE HERMITE-PADÉ TABLE

by

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No. 49 May, 1989.

Abstract

This paper investigates the structure and degeneracy in the table of quadratic Hermite-Padé forms. In §2 it is noted that a space of quadratic Hermite-Padé forms for $f(x)$ may be multi-dimensional and the optimal choice of a representative form is discussed. In §3 the structure of the table of quadratic Hermite-Padé forms is considered. The Padé case is considered first and it is suggested that a D-table of degenerate Padé forms is perhaps a better indication of the structure than the more traditional C-table. The usual table of Padé approximants may be deduced from the D-table in much the same way as it is from the C-table. An analagous structure is deduced for the D-table of quadratic Hermite-Padé forms. This idea can be applied to general Hermite-Padé forms.

AMS Classification: 41A21, 41A30
Key words and phrases: Hermite-Padé approximation, Padé approximation, quadratic function approximation.

1. Introduction

Firstly, the definition of Hermite-Padé forms and some notation are introduced:

(i) Let $f(x)$ be a function, analytic in a neighbourhood of the origin whose power series expansion about the origin is known.

(ii) Let $g_0, g_1, \ldots, g_n$ be functions such that $\forall i \in \{0, \ldots, n\}$ $g_i(f(x))$ is analytic in a neighbourhood of the origin and its power series expansion about the origin is known.

(iii) Let $A_0, A_1, \ldots, A_n \in \mathrm{N} \cup \{-1\}$ and $N = \sum_{i=0}^{n} A_i$.

(iv) Let $a_0(x), a_1(x), \ldots, a_n(x)$ be polynomials in x with $\deg(a_i(x)) \leq A_i$ $\forall i \in \{0, \ldots, n\}$, such that

$$\sum_{i=0}^{n} a_i(x)\, g_i(f(x)) = O\left(x^{N+n}\right) \tag{1}$$

(The only polynomial of degree -1 is, by definition, the zero polynomial.)

Note that such $a_i(x)$, not all zero, must exist since (1) represents a homogeneous system of $N+n$ linear equations in the $N+n+1$ unknown coefficients of the $a_i(x)$.

Let $g_i(f(x)) = \sum_{j=0}^{\infty} g_i^j x^j$, $\quad a_i(x) = \sum_{j=0}^{A_i} a_i^j x^j$.

Then the system of linear equations given by (1) can be represented as the following matrix equation:

$$\begin{bmatrix} g_0^0 & 0 & 0 & & & g_n^0 & 0 & 0 & \\ g_0^1 & g_0^0 & 0 & & & g_n^1 & & 0 & \\ g_0^2 & g_0^1 & & & & & & & \\ & & & g_0^0 & & & & & g_n^0 \\ \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & \\ g_0^{N+n-1} & g_0^{N+n-2} & \cdots & g^{N+n-A_0-1} & & g_n^{N+n-1} & & & g^{N+n-A_n-1} \end{bmatrix} \begin{bmatrix} a_0^0 \\ a_0^1 \\ \vdots \\ a_0^{A_0} \\ \vdots \\ \\ a_k^{A_k} \end{bmatrix} = 0 \tag{2}$$

A set of $a_i(x)$ derived in this way is known as a $(A_n, A_{n-1}, \ldots, A_0)$ Hermite-Padé form for the system $g_i(f(x))$. Such a form will be represented as $\{a_n(x), ..., a_0(x)\}$ or as $\sum_{i=0}^{n} a_i(x) g_i(y)$. The set of all $(A_n, A_{n-1}, \ldots, A_0)$ forms for $g_i(f(x))$ forms a linear space for each set of $A_n, ..., A_0$. Set

$$\sum_{i=0}^{n} a_i(x)\, g_i(y(x)) = 0. \tag{3}$$

It is the solution of (3) for $y(x)$ (not necessarily unique) that gives a Hermite-Padé approximation for $f(x)$.

This is the standard way of defining Hermite-Padé approximations. See, for example, Della Dora and Di Crescenzo [4], Paszkowski [5] and Baker and Lubinsky [2] in which there appears an extensive bibliography.

Examples

(i) Setting $g_i(f(x)) = \dfrac{d^i f(x)}{dx^i}$ gives the differential Hermite-Padé approximants (see [2]). The case $n = 1$ is that of the Baker $D-$log approximation (see [1]).

(ii) Setting $g_i(f(x)) = f(x)^i$ gives the algebraic Hermite-Padé approximants of which there are several important special cases:

1. $n = 1, A_1 = 0$ gives the Taylor approximation.
2. $n = 1$ gives the Padé approximation.
3. $n = 2$ gives the quadratic approximation. (again, see [2] and the references therein)

Attention is now restricted to Padé and quadratic forms. Generalisations to other Hermite-Padé forms are obvious.

2. An optimal choice from the space of (A_2, A_1, A_0) forms

Let $A_2, A_1, A_0 \in$ N. In general, the space of (A_2, A_1, A_0) forms for $f(x)$ may be multi-dimensional. This corresponds to the matrix appearing in equation (2) having rank < N+2. If the rank N+3–k then the space of forms has dimension k. If k>1 there is a choice of which solution to take. In the context of a sequence of quadratic Hermite-Padé forms the results of [3] indicate that it is not too important which solution is chosen. If, however, a basis for the solution space is known it is clearly desirable to have some method of choosing a form of maximal order from this space. That is, we seek a one dimensional subspace whose elements satisfy $\sum a_i(x) f(x)^i = O\left(x^M\right)$ where M is maximal over the space of (A_2, A_1, A_0) forms. A procedure for finding such a subspace is now given.

Let S_{k_1} be the space of all (A_2, A_1, A_0) forms for $f(x)$ and let $\{m^{i,w} : i \in \{1, ..., k_w\}\}$ be a basis for the k_w dimensional space S_{k_w} (to be defined below) of (A_2, A_1, A_0) forms.
Let $\left\{a_j^{i,w}(x) : j \in \{0, 1, 2\}\right\}$ be the polynomial coefficients of $m^{i,w}$ and
let $\left\{e_j^{i,w} : j \in \{0, 1, 2, ...\}\right\}$ be the coefficients of the power series $\sum_{j=0}^{\infty} e_j^{i,w} x^j = \sum_{j=0}^{2} a_j^{i,w}(x) f(x)^j$

Step 1 : Set $w = 1$

Step 2 : If $k_w = 1$:

then the one dimensional subspace has been found so terminate.

Else if $\sum e_j^{i,w} x^j \equiv 0$ for some i

then $f(x)$ has been found exactly so terminate.

Else set $S_{k_{w+1}} = \left\{ \sum_{i=1}^{k_w} c_i m^{i,w} : \sum_{i=1}^{k_w} c_i e_{N+1+w}^{i,w} = 0 \right\}$

Step 3 : Set $w = w + 1$.

Go to Step 2.

Note that:

(i) $S_{k_{w+1}}$ is clearly a linear subspace of S_{k_w} (so $k_{w+1} \leq k_w$) and it is non-trivial since $\sum_{i=1}^{k_w} c_i e_{N+1+w}^{i,w} = 0$ is a linear homogeneous equation in the k_w (≥ 2) unknowns c_i.

(ii) This process must terminate since if $k_w > 1 \;\; \forall w \in \mathrm{N}$ then $\exists p \in \mathrm{N}$ such that $\sum_{p=0}^{\infty} e_j^{1,p} x^j \equiv 0$.

(iii) Since this procedure has raised the order of the form (A_2, A_1, A_0) by w after w iterations, it follows that $m^{i,w}$ is also a form of types

$$\{(A_2 + r, A_1 + s, A_0 + t) \; : \; r, s, t \in \mathbf{Z}^+, \; r + s + t \leq w - 1\}.$$

One common example of when this may occur is when $f(x)$ is an even function. If $f(x)$ is even then so is $f(x)^2$, and an examination of the matrix appearing equation (2) in this case reveals that if the A_k are all even then the matrix has rank of at most $N + 1$. Hence, in this case the solution space for an even function has a dimension of at least two. This can be observed in the tables of quadratic forms for $\cos(x)$ (see Tables 1 and 3).

Example.

A basis for the $(0,2,0)$ forms for $\cos(x)$ is given by (see the following table):
$m^{1,1} = \left(x^2+2\right)y-2, \quad e_4^{1,1} = -5/12$
$m^{2,1} = y^2-2y+1 \qquad e_4^{2,1} = 1/4$
So $m^{1,2} = 3m^{1,1}+5m^{2,1} = 5y^2+(3x^2-4)y-1$ (with $5f(x)^2+(3x^2-4)f(x)-1 = O(x^5)$). Note that this means that $m^{1,2}$ is a form of types $(1,2,0),(0,3,0)$ and $(0,2,1)$. This can be seen in Table 1.
In fact $5f(x)^2+(3x^2-4)f(x)-1 = O(x^6)$ so it is also a form of other types also.

Table 1: Hermite-Padé forms for $\cos x$

$A_2 = 0$

A_0 \ A_1	0	1	2
0	$a_0(x) = -1$ $a_1(x) = 1$ $a_2(x) = 0$	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = -2$ $a_1(x) = x^2+2$ $a_2(x) = 0$
	$a_0(x) = -1$ $a_1(x) = 0$ $a_2(x) = 1$		$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$
1	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = -1$ $a_1(x) = 3x^2-4$ $a_2(x) = 5$
2	$a_0(x) = x^2-2$ $a_1(x) = 2$ $a_2(x) = 0$	$a_0(x) = -3x^2+7$ $a_1(x) = -8$ $a_2(x) = 1$	$a_0(x) = 5x^2-12$ $a_1(x) = x^2+12$ $a_2(x) = 0$
	$a_0(x) = x^2-1$ $a_1(x) = 0$ $a_2(x) = 1$		$a_0(x) = -3x^2+7$ $a_1(x) = -8$ $a_2(x) = 1$

$A_2 = 1$

A_1 / A_0	0	1	2
0	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = -1$ $a_1(x) = 3x^2 - 4$ $a_2(x) = 5$
1	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = x$ $a_1(x) = -2x$ $a_2(x) = x$	$a_0(x) = -1$ $a_1(x) = 3x^2 - 4$ $a_2(x) = 5$
2	$a_0(x) = -3x^2 + 7$ $a_1(x) = -8$ $a_2(x) = 1$	$a_0(x) = -3x^2 + 7$ $a_1(x) = -8$ $a_2(x) = 1$	$a_0(x) = 11x^2 - 27$ $a_1(x) = 4x^2 + 24$ $a_2(x) = 3$

$A_2 = 2$

A_1 / A_0	0	1	2
0	$a_0(x) = 1$ $a_1(x) = -2$ $a_2(x) = 1$	$a_0(x) = 5$ $a_1(x) = -16$ $a_2(x) = 3x^2 + 11$	$a_0(x) = -1$ $a_1(x) = 3x^2 - 4$ $a_2(x) = 5$
	$a_0(x) = -2$ $a_1(x) = 2$ $a_2(x) = x^2$		$a_0(x) = 12$ $a_1(x) = -11x^2 - 12$ $a_2(x) = 5x^2$
1	$a_0(x) = 5$ $a_1(x) = -16$ $a_2(x) = 3x^2 + 11$	$a_0(x) = 5$ $a_1(x) = -16$ $a_2(x) = 3x^2 + 11$	$a_0(x) = -3$ $a_1(x) = 64x^2 - 144$ $a_2(x) = 11x^2 + 147$
2	$a_0(x) = -3x^2 + 7$ $a_1(x) = -8$ $a_2(x) = 1$	$a_0(x) = -16x^2 + 39$ $a_1(x) = -48$ $a_2(x) = x^2 + 9$	$a_0(x) = 11x^2 - 27$ $a_1(x) = 4x^2 + 24$ $a_2(x) = 3$
	$a_0(x) = 11x^2 - 24$ $a_1(x) = 24$ $a_2(x) = x^2$		$a_0(x) = -49x^2 + 120$ $a_1(x) = -12x^2 - 120$ $a_2(x) = x^2$

3. Structure and degeneracy in the table of quadratic Hermite-Padé forms.

The Padé table

To provide some indication of possible generalisations to the quadratic case we first give some results concerning the Padé table. A D-table showing the degenerate Padé forms is introduced instead of the more traditional C-table (see [1]). This table gives more information about the structure and degeneracy of the Padé forms and leads directly to the Padé table of approximants in much the same way as the C-table. These results (and parts of the proofs thereof) are adapted from Baker [1].

Let $\{m_i : i \in I\}$ be the linear space of (A_1, A_0) Padé forms (for $f(x)$). Let $m_i = a_1^i(x)y + a_0^i(x)$. Note that if $a_1^i(0) = 0$ then $a_0^i(0) = 0$ (since $a_1^i(x)f(x) + a_0^i(x) = O(x^{A_1+A_0+1})$). Let $r_i = \max\{r \in \mathrm{N} : x^r | a_j^i(x) \quad \forall j \in \{0,1\}\}$ and let $a_j^i(x) = x^{r_i} b_j^i(x)$. Then $y(x) = \dfrac{-b_0^i(x)}{b_1^i(x)} = f(x) + O\left(x^{A_1+A_0-r_i+1}\right)$ and $b_1^i(0) \neq 0$.

Theorem 1. (Uniqueness) (see Baker [1] Theorem 1.1).
If $y_i(x)$ and $y_j(x)$ are the Padé approximants corresponding to the (A_1, A_0) Padé forms m_i, m_j, then

$$y_i(x) = y_j(x) \quad \forall i, j \in I .$$

Proof.

$$\frac{b_0^i(x)}{b_1^i(x)} - \frac{b_0^j(x)}{b_1^j(x)} = O\left(x^{A_1+A_0+1-max\{r_i,r_j\}}\right)$$

$$\Rightarrow b_0^i(x)\, b_1^j(x) - b_0^j(x)\, b_1^i(x) = O\left(x^{A_1+A_0+1-\max\{r_i,r_j\}}\right) \tag{4}$$

But the left hand side of (4) is a polynomial of degree at most $A_0 - r_i + A_1 - r_j < A_1 + A_0 + 1 - \max\{r_i, r_j\}$ thus is identically zero. Hence

$$\frac{b_0^i(x)}{b_1^i(x)} \equiv \frac{b_0^j(x)}{b_1^j(x)}$$

and it follows that all (A_1, A_0) Padé forms are just polynomial multiples of some basic form.

□

From Theorem 1, one can choose a unique (up to a constant factor) representative of the (A_1, A_0) forms, i.e. a form of minimal degree from those of maximal order.

Note that :

(i) $p(A_1, A_0)$ has no common polynomial factor except possibly $x^j, j \in \mathrm{N}$.

(ii) If $D(A_1, A_0) < \infty$ then one of the coefficients of $p(A_1, A_0)$ must have full degree.

(iii) In the case $D(A_1, A_0) < \infty$ the conditions (i) and (ii) above completely characterise $p(A_1, A_0)$.

Let $R(m_i)(x) = \sum_{j=0}^{1} a_j^i(x) f(x)^j$, for $m_i \in S$, the linear space of (A_1, A_0) forms.

Definition 1 :The unique representative, $p(A_1, A_0)$ of the (A_1, A_0) forms is defined by

$$p(A_1, A_0) = \Big\{ m_i : \sum_j \deg\left(a_j^i(x)\right) = \min\Big\{ \sum_j \deg(a_j^k(x)) \ : O\left(R(m_k)(x)\right) \\ = \max\Big\{ O\left(R(m_r)(x)\right) : r \in I \Big\} \Big\} \Big\}$$

Definition 2 :The degeneracy, $D(A_1, A_0)$ of the Padé form $p(A_1, A_0)$ is defined by

$$D(A_1, A_2) = \mathrm{Ord}\left(R\left(p(A_1, A_2)\right)(x)\right) - (A_1 + A_0 + 1)$$

where $\mathrm{Ord}(R(x)) = N$ if $O(R(x)) = O(x^N), \neq O(x^{N+1})$ as $x \to \infty$.

The degeneracy, D, is the amount of extra matching obtained from $p(A_1, A_0)$. Trefethen [6] introduced a similar concept for the Padé approximation rather than the Padé form, when studying a related problem. However this definition and presentation makes these concepts clearer. The two dimensional table of D values, the D-table, will be shown to give the block structure of the Padé table in a somewhat easier fashion than the C-table (see Baker [1], p13).

Theorem 2. Let $D(A_1, A_0) < \infty$. Then

$$p(A_1, A_0) = x^r p(A_1 - r, A_0 - r), r \in \mathrm{N} \Leftrightarrow D(A_1 - r, A_0 - r) = D(A_1, A_0) + r.$$

Proof. Note that $D(A_1 - r, A_0 - r) = D(A_1, A_0) + r$

$$\Leftrightarrow \mathrm{Ord}\left(R\left(p(A_1, A_0)\right)(x)\right) = \mathrm{Ord}\left(R\left(p(A_1 - r, A_0 - r)\right)(x)\right) + r.$$

If $p(A_1, A_0) = x^r p(A_1 - r, A_0 - r)$ then clearly $D(A_1 - r, A_0 - r) = D(A_1, A_0) + r$.
If $D(A_1 - r, A_0 - r) = D(A_1, A_0) + r$ then

$$\mathrm{Ord}\left(R\left(p(A_1, A_0)\right)(x)\right) = \mathrm{Ord}\left(R\left(p(A_1 - r, A_0 - r)\right)(x)\right) + r.$$

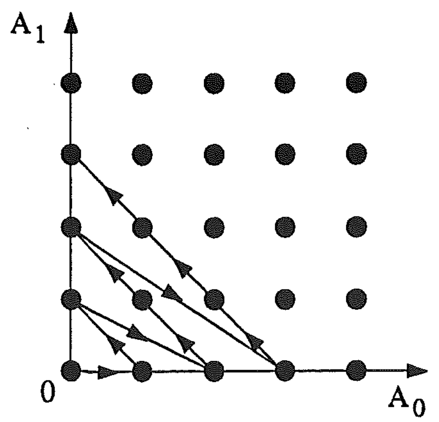
Certainly $\mathrm{Ord}\left(R\left(x^r p(A_1 - r, A_0 - r)\right)(x)\right) = \mathrm{Ord}\left(R\left(p(A_1, A_0)\right)(x)\right)$. Since $x^r p(A_1 - r, A_0 - r)$ has no common factor except $x^s, s \in \mathrm{N}$ then $p(A_1, A_0) = x^r p(A_1 - r, A_0 - r)$. □

It is obvious that $p(A_1, A_0)$ yields an approximation

$$y(x) = f(x) + O\left(x^{A_1+A_0-r+D(A_1,A_0)+1}\right) \tag{5}$$

where x^r is the maximal factor of $p(A_1, A_0)$. So the only "problems" in the Padé table occur when $r > D(A_1, A_0)$. In short, using Theorem 2, a form gives an approximation with less than expected order of accuracy if and only if some appropriate earlier form has $D > 0$.

Now set out the Padé table and search through forms in the order shown (this is similar to Baker's proof of the structure of the Padé table ([1], p17)).



Note in passing that the forms on the borders of the table can easily be shown to have no factor of x. Suppose $p(A_1, A_0)$ is the first form found with $D(A_1, A_0) = t > 0$. The table is "normal" (see Baker [1], p24) up to this point. $p(A_1, A_0)$ generates a block structure (we take for example $t = 3$). In the table that follows $[p(A_1, A_0), D]$ will denote $p(A_1, A_0)$ with $D(A_1, A_0) = D$. This table is rotated 90 degrees from the traditional forms in Baker [1]. Note how the equal D values lie rows and columns bordering the initial element as in the diagram below.

0	0	0	0
1	1	1	0
2	2	1	0
3	2	1	0

$[p(A_1+4, A_0), D(A_1+4, A_0)]$	$[p(A_1+4, A_0+1), D(A_1+4, A_0+1)]$	$[p(A_1+4, A_0+2), D(A_1+4, A_0+2)]$	$[p(A_1+4, A_0+3), D(A_1+4, A_0+3)]$	$[p(A_1+4, A_0+4), D(A_1+4, A_0+4)]$
$[p(A_1, A_0), 0]$	$[xp(A_1, A_0), 0]$	$[x^2p(A_1, A_0), 0]$	$[x^3p(A_1, A_0), 0]$	$[p(A_1+3, A_0+4), D(A_1+3, A_0+4)]$
$[p(A_1, A_0), 1]$	$[xp(A_1, A_0), 1]$	$[x^2p(A_1, A_0), 1]$	$[x^2p(A_1, A_0), 0]$	$[p(A_1+2, A_0+4), D(A_1+2, A_0+4)]$
$[p(A_1, A_0), 2]$	$[xp(A_1, A_0), 2]$	$[xp(A_1, A_0), 1]$	$[xp(A_1, A_0), 0]$	$[p(A_1+1, A_0+4), D(A_1+1, A_0+4)]$
$[p(A_1, A_0), 3]$	$[p(A_1, A_0), 2]$	$[p(A, A_0), 1]$	$[p(A_1, A_0), 0]$	$[p(A_1, A_0+4, D(A_1, A_0+4)]$

Table 1

If $a_1(x)$ and $a_0(x)$ are the polynomial coefficients of $p(A_1, A_0)$ then since $D(A_1-1, A_0) = D(A_1, A_0-1) = 0$ it follows that $\deg(a_j(x)) = A_j$. Also, $p(A_1, A_0)$ does not have a factor of x since, by assumption, $D(A_1-1, A_0-1) = 0$. The entries in the table are now justified.

(i) The entries $\{(A_1+i, A_0+j) : i, j \in \{0,1,2,3\}\}$:
For a given (i, j) the entry is easily seen to be a (A_1+i, A_0+j) form with no common factor except possibly $x^j, j \in \mathrm{N}$. Also one of the coefficients of the entry has maximum degree so this entry equals $p(A_1+i, A_0+j)$ (by note (iii) after Theorem 1).

(ii) The top and right borders of entries do not have a factor of x:
Take for example (A_1+4, A_0) and suppose otherwise. Then $p(A_1+4, A_0) = xp(A_1+3, A_0-1)$ and $D(A_1+3, A_0-1) > 0$. Since $p(A_1+3, A_0-1)$ has no common factor except possibly $x^t, t \in \mathbf{Z}^+$ then $p(A_1+3, A_0) = x^j p(A_1+3, A_0-1), j \in \mathbf{Z}^+$. But $p(A_1+3, A_0) = p(A_1, A_0)$ and $p(A_1, A_0)$ has no factor of x so $p(A_1+3, A_0-1) = p(A_1, A_0)$. Hence $\deg(a_0(x)) < A_0$ a contradiction.

The final structure is that of a 3×3 block of forms which have $x^r, r \in \mathrm{N}$ as a factor. The forms $\{(A_1+1+i, A_0+1+j) : i+j \leq D(A_1, A_0)-2, \quad i, j \in \mathbf{Z}^+\}$ still yield an approximation of full order whilst the others in the block do not. In Baker's terminology (Baker [1]) these latter approximations are said "not to exist" although clearly they do, but with a less than expected order of accuracy. A Padé table having such blocks is said to be "non-normal".

The concept of the D-table outlined above, generalises easily to give the standard theorem on the structure of the Padé table approximations. The version quoted below is a modification of that from Baker [1,Theorem 2.3].

Theorem 3. The Padé table can be completely dissected into $r \times r$ blocks with horizontal and vertical sides, $r \geq 1$. Let $[\lambda/\mu]$ denote the unique ($\lambda + \mu$ = minimum) member of a particular $r \times r$ block. Then:

(i) The $[\lambda/\mu]$ exists and the numerator and denominator are of full nominal degree.

(ii) $[\lambda + p/\mu + q] = [\lambda/\mu]$ for $p + q \leq r-1, p \geq 0, q \geq 0$.

(iii) $[\lambda + p/\mu + q] = [\lambda/\mu]$ for $p + q \geq r, r-1 \geq p \geq 1, r-1 \geq q \geq 1$, but as an approximation of less than full order, given by (5).

(iv) $C(\lambda + p/\mu + q) = 0$ for $1 \leq p \leq r-1, 1 \leq q \leq r-1$ and $C \neq 0$ otherwise ($C(\lambda + p/\mu + q)$ is $a_1(0)$ in our notation).

The quadratic Hermite-Padé table

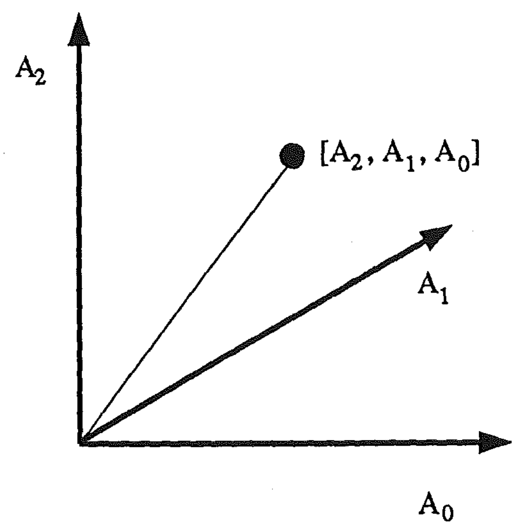
In the previous section it was shown that the Padé table has a simple structure and one would hope that similar theorems could be formulated in the quadratic case. The basic property which underlies the Padé structure is that of Theorem 1, i.e. that any two (A_1, A_0) Padé forms differ by at most a polynomial factor. This is *not*, unfortunately, true for quadratic forms as can be seen by the example in §2. Hence, although, as will be shown, a table of D values gives much valuable information, complications may arise and break down any systematic structure.

Basic degenerate structures

If a form has a degeneracy $D > 0$ then this propagates to the forms around it in a manner similar to that in the Padé table. Examples are given below, the various entries being deduced from simple order matching principles. It should be realised that, lacking any kind of uniqueness theorem, each entry is *one* (i, j, k) form only. There may be others, which may be of different degeneracy. Unless a type of uniqueness result can be found it is expected that the general case will involve overlapping of the following structures. This approach does, however, completely explain the structure of many of the previous examples.

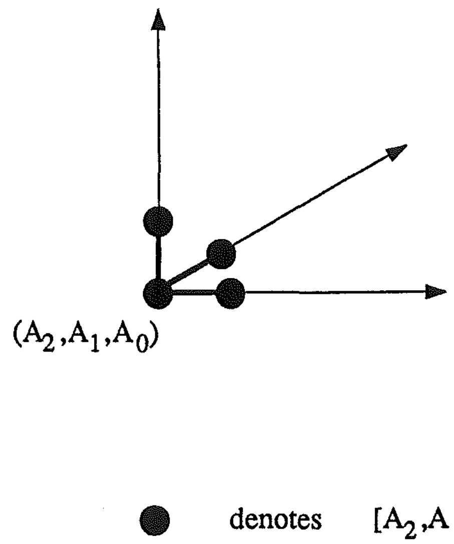
Examples

The following $3-D$ representation is used:



Example 1

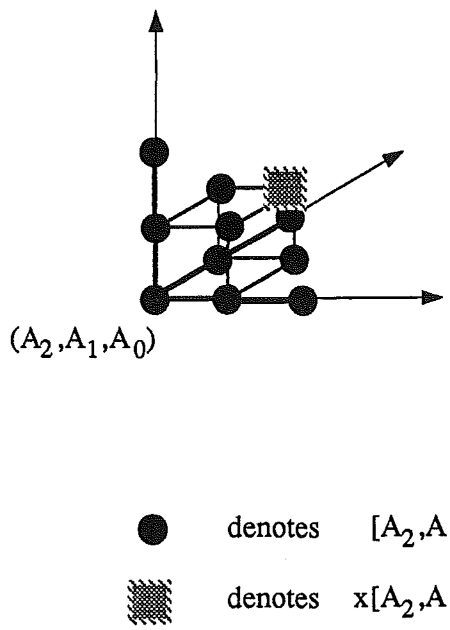
$D(A_2, A_1, A_0) \; = \; 1$



For instance, see the example at the end of §2 where $5y^2+(3x^2-4)y-1$ is a (0,2,0) form for $\cos(x)$ giving this structure, i.e. the (0,2,0) form is also a form of types (1,2,0), (0,3,0), (0,2,1).

Example 2

$D(A_2, A_1, A_0) \; = \; 2$



The (0,0,0) form for $\cos(x)$, that is $y^2 - 2y + 2$, gives this structure (see Table 1).

A further example is given in the table following. Taking simple linear combinations of the basis elements, the (2,2,2) form, $(23x^2+465)y^2+(344x^2+960)y+578x^2-1425$, can be seen to generate this structure; yet overlapping it starting at (2,2,4) is another such structure generated by the form $(14x^2+417)y^2+(704x^2+11136)y-237x^4+5267x^2-11553$ (see the comments immediately preceding this section of examples).

Table 3: Hermite-Padé forms for cos(x)

$A_2 = 2$

A_1 / A_0	2	3	4
2	$a_0(x)=11x^2-27$ $a_1(x)=4x^2+24$ $a_2(x)=3$	$a_0(x)=578x^2-1425$ $a_1(x)=344x^2+960$ $a_2(x)=23x^2+465$	$a_0(x)=-723x^2+1783$ $a_1(x)=-23x^4-12x^2-2096$ $a_2(x)=313$
	$a_0(x)=-49x^2+120$ $a_1(x)=-12x^2-120$ $a_2(x)=x^2$		$a_0(x)=22483x^2-55440$ $a_1(x)=465x^4+4924x^2+55440$ $a_2(x)=313x^2$
3	$a_0(x)=578x^2-1425$ $a_1(x)=344x^2+960$ $a_2(x)=23x^2+465$	$a_0(x)=578x^2-1425$ $a_1(x)=344x^2+960$ $a_2(x)=23x^2+465$	$a_0(x)=-5828x^2+14379$ $a_1(x)=-474x^4+5152x^2-28128$ $a_2(x)=361x^2+13749$
4	$a_0(x)=-23x^4+477x^2-1037$ $a_1(x)=48x^2+1024$ $a_2(x)=13$	$a_0(x)=-237x^4+5267x^2-11553$ $a_1(x)=704x^2+11136$ $a_2(x)=14x^2+417$	$a_0(x)=361x^4-8367x^2+18447$ $a_1(x)=-28x^4-768x^2-18624$ $a_2(x)=177$
	$a_0(x)=465x^4-9317x^2+20160$ $a_1(x)=-776x^2-20160$ $a_2(x)=13x^2$		$a_0(x)=-4583x^4+105269x^2-231840$ $a_1(x)=278x^4+10592x^2+231840$ $a_2(x)=59x^2$

$A_2 = 3$

A_1 / A_0	2	3	4
2	$a_0(x) = 578x^2 - 1425$ $a_1(x) = 344x^2 + 960$ $a_2(x) = 23x^2 + 465$	$a_0(x) = 578x^2 - 1425$ $a_1(x) = 344x^2 + 960$ $a_2(x) = 23x^2 + 465$	$a_0(x) = -5828x^2 + 14379$ $a_1(x) = -474x^4 + 5152x^2 - 28128$ $a_2(x) = 361x^2 + 13749$
3	$a_0(x) = 578x^2 - 1425$ $a_1(x) = 344x^2 + 960$ $a_2(x) = 23x^2 + 465$	$a_0(x) = 578x^3 - 1425x$ $a_1(x) = 344x^3 + 960x$ $a_2(x) = 23x^3 + 465x$	$a_0(x) = -5828x^2 + 14379$ $a_1(x) = -474x^4 + 5152x^2 - 28128$ $a_2(x) = 361x^2 + 13749$
4	$a_0(x) = -237x^4 + 5267x^2 - 11553$ $a_1(x) = 704x^2 + 11136$ $a_2(x) = 14x^2 + 417$	$a_0(x) = -237x^4 + 5267x^2 - 11553$ $a_1(x) = 704x^2 + 11136$ $a_2(x) = 14x^2 + 417$	$a_0(x) = 17911x^4 - 427753x^2 + 946395$ $a_1(x) = -2416x^4 - 26944x^2 - 984960$ $a_2(x) = 782x^2 + 38565$

$A_2 = 4$

A_1 / A_0	2	3	4
2	$a_0(x) = 578x^2 - 1425$ $a_1(x) = 344x^2 + 960$ $a_2(x) = 23x^2 + 465$	$a_0(x) = 74095x^2 - 182802$ $a_1(x) = 66112x^2 + 69504$ $a_2(x) = 237x^4 + 7843x^2 + 113298$	$a_0(x) = -5828x^2 + 14379$ $a_1(x) = -474x^4 + 5152x^2 - 28128$ $a_2(x) = 361x^2 + 13749$
	$a_0(x) = -519x^2 + 1279$ $a_1(x) = -216x^2 - 1088$ $a_2(x) = x^4 - 191$		$a_0(x) = 305727x^2 - 754287$ $a_1(x) = 15686x^4 - 69792x^2 + 1036704$ $a_2(x) = 361x^4 - 282417$
3	$a_0(x) = 74095x^2 - 182802$ $a_1(x) = 66112x^2 + 69504$ $a_2(x) = 237x^4 + 7843x^2 + 113298$	$a_0(x) = 74095x^2 - 182802$ $a_1(x) = 66112x^2 + 69504$ $a_2(x) = 237x^4 + 7843x^2 + 113298$	$a_0(x) = -3571907x^2 + 8813250$ $a_1(x) = -745936x^4 + 13104064x^2 - 39012480$ $a_2(x) = 17911x^4 + 1160833x^2 + 30199230$
4	$a_0(x) = -237x^4 + 5267x^2 - 11553$ $a_1(x) = 704x^2 + 11136$ $a_2(x) = 14x^2 + 417$	$a_0(x) = -46621x^4 + 1083296x^2 - 2389095$ $a_1(x) = 180608x^2 + 2234880$ $a_2(x) = 151x^4 + 7751x^2 + 154215$	$a_0(x) = 17911x^4 - 427753x^2 + 946395$ $a_1(x) = -2416x^4 - 26944x^2 - 984960$ $a_2(x) = 782x^2 + 38565$
	$a_0(x) = 7843x^4 - 169923x^2 + 371523$ $a_1(x) = -19392x^2 - 364416$ $a_2(x) = 14x^4 - 7107$		$a_0(x) = -50471x^4 + 1198575x^2 - 2650095$ $a_1(x) = 5392x^4 + 100800x^2 + 2701440$ $a_2(x) = 34x^4 - 51345$

These results may be summarised in the following theorem.

Theorem 4. If a (A_2, A_1, A_0) form has degeneracy $D(A_2, A_1, A_0) = t \in \mathbf{N}$ then :

$$x^r [A_2, A_1, A_0], \quad r \epsilon \left\{0, 1, ..., max\left\{s : s \leq t/2, s \epsilon \mathbf{Z}^+\right\}\right\}$$

is a form of type

$$\left\{(A_2 + r + i_r, A_1 + r + j_r, A_0 + r + k_r) : i_r + j_r + k_r \leq t - 2r, \quad i_r, j_r, k_r \epsilon \mathbf{Z}^+\right\}$$

with

$$D\left(A_2 + r + i_r, A_1 + r + j_r, A_0 + r + k_r\right) = t - 2r - (i_r + j_r + k_r) .$$

Proof. Since

$$R\left([A_2, A_1, A_0]\right)(x) = O\left(x^{A_2 + A_1 + A_0 + 2 + t}\right)$$

then

$$R\left(x^r [A_2, A_1, A_0]\right)(x) = O\left(x^{A_2 + A_1 + A_0 + 2 + t + r}\right)$$

Hence $x^r[A_2, A_1, A_0]$ will be a quadratic form of type $(A_2 + r, A_1 + r, A_0 + r)$ provided that $r \leq t/2$. The degeneracy of this form is $t - 2r$. Further, it follows, as in §2, that this form is also a form of the type

$$\left\{(A_2 + r + i_r, A_1 + r + j_r, A_0 + r + k_r) : i_r + j_r + k_r \leq t - 2r, \quad i_r, j_r, k_r \epsilon \mathbf{Z}^+\right\}$$

with

$$D\left(A_2 + r + i_r, A_1 + r + j_r, A_0 + r + k_r\right) = t - 2r - (i_r + j_r + k_r) .$$

□

Note also that there is a direct connection between k, the dimension of the solution space for (A_2, A_1, A_0) forms and the degeneracy, D, of the optimal (see §2) (A_2, A_1, A_0) form, namely $D \geq k - 1$. The example $f(x) = \cos(x), (A_2, A_1, A_0) = (0, 0, 0)$ (see Table 1) shows that equality need not hold.

4. Conclusion

In this paper, two main topics have been explored :

(i) In §2 it was shown that there is a 1 dimensional subspace of each space of (A_2, A_1, A_0) forms whose elements are of maximal order. This, of course, applies to all Hermite-Padé systems. A form of maximal order does not necessarily yield an approximation of the same order, as can be seen by the example of the (0,0,0) approximation to $\cos(x)$.

(ii) In §3 the question of the structure of the table of quadratic forms was examined. Firstly we introduced the concept of the D-table of Padé forms and deduced the well-known theorem on the block structure of the Padé table. This idea was applied to the D-table for quadratic forms, and whilst not completely solving the problem, because of the complication of overlapping structures, it was shown to indicate the sorts of structures which occur.

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